

# DERIVATIONS INTO DUALS OF CLOSED IDEALS OF BANACH ALGEBRAS

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**ABSTRACT.** Let  $\mathcal{A}$  be a Banach algebra. We study those closed ideals  $I$  of  $\mathcal{A}$  for which the first cohomology group of  $\mathcal{A}$  with coefficients in  $I^*$  is trivial; i.e.  $H^1(\mathcal{A}, I^*) = \{0\}$ . We investigate such closed ideals when  $\mathcal{A}$  is weakly amenable or biflat. Also we give some hereditary properties of ideal amenability.

## 1. INTRODUCTION

Let  $\mathcal{A}$  be a Banach algebra and  $X$  a Banach  $\mathcal{A}$ -bimodule. Then  $X^*$ , the dual space of  $X$ , is also a Banach  $\mathcal{A}$ -bimodule with module multiplications defined by

$$\langle x, a.x^* \rangle = \langle x.a, x^* \rangle, \quad \langle x, x^*.a \rangle = \langle a.x, x^* \rangle, \quad (a \in \mathcal{A}, x \in X, x^* \in X^*).$$

In particular,  $I$  and  $I^*$  are Banach  $\mathcal{A}$ -bimodule for every closed ideal  $I$  of  $\mathcal{A}$ . A derivation from  $\mathcal{A}$  into  $X$  is a continuous linear operator  $D$  such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

We define  $\delta_x(a) = a \cdot x - x \cdot a$ ; for each  $x \in X$  and  $a \in \mathcal{A}$ .  $\delta_x$  is a derivation from  $\mathcal{A}$  into  $X$ , which is called inner derivation. A Banach algebra  $\mathcal{A}$  is amenable if every derivation from  $\mathcal{A}$  into every dual  $\mathcal{A}$ -bimodule  $X^*$  is inner i.e.  $H^1(\mathcal{A}, X^*) = \{0\}$ . This definition was introduced by B. E. Johnson in [J1], [Run1] and [He]. A Banach algebra  $\mathcal{A}$  is weakly amenable if every derivation from  $\mathcal{A}$  into  $\mathcal{A}^*$  is inner i.e.  $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$ . Bade, Curtis and Dales [B-C-D] have introduced the concept of weak amenability for commutative Banach algebras. You can see also [J3], [D-Gh], [G1], [G2] and [G3]. Let  $n \in \mathbb{N}$ , a Banach algebra  $\mathcal{A}$  is called  $n$ -weakly amenable if  $H^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$ , where  $\mathcal{A}^{(n)}$  is the  $n$ -th dual of  $\mathcal{A}$  [D-Gh-G] and [J2].

Let  $G = SL(2, \mathbb{R})$ , the set of elements in  $M_2(\mathbb{R})$  with determinant 1, also let  $\mathcal{A} = L^1(G)$  and  $I$  be the augmentation ideal of  $\mathcal{A}$ , then theorem 5.2 of [J-W] implies that  $H^1(\mathcal{A}, I^*) \neq \{0\}$ . On the other hand  $\mathcal{A}$  is weakly amenable. This example guides us to the following definitions.

Let  $\mathcal{A}$  be a Banach algebra and  $I$  be a closed two-sided ideal of  $\mathcal{A}$ , then  $\mathcal{A}$  is  $I$ -weakly amenable if every derivation from  $\mathcal{A}$  into  $I^*$  is inner, in other words  $H^1(\mathcal{A}, I^*) = \{0\}$ . We call  $\mathcal{A}$  ideally

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2000 *Mathematics Subject Classification.* Primary 46H25, 16E40.

*Key words and phrases.* Derivation, weakly amenabl, biflat, ideally amenable.

amenable if  $\mathcal{A}$  is  $I$ -weakly amenable for every closed ideal  $I$  of  $\mathcal{A}$  [E-Y], [E-H]. Let  $n \in \mathbb{N}$ , a Banach algebra  $\mathcal{A}$  is called  $n$ -ideally amenable if for every closed two-sided ideal  $I$  in  $\mathcal{A}$ ,  $H^1(\mathcal{A}, I^{(n)}) = \{0\}$ .

Obviously, an ideally amenable Banach algebra is weakly amenable. Since every closed ideal of  $\mathcal{A}$  is a Banach  $\mathcal{A}$ -bimodule, then an amenable Banach algebra is ideally amenable. There are some examples of Banach algebras to show that ideal amenability is not equivalent to weak amenability or amenability. In the following we give some of them.

1- Let  $\mathcal{A}$  be the unitization of the augmentation ideal of  $L^1(SL(2, \mathbb{R}))$ . Then  $\mathcal{A}$  is weakly amenable and  $\mathcal{A}$  is not ideally amenable [E-Y].

2- Let  $\mathcal{A} = L^1(SL(2, \mathbb{R}))$ . Then  $\mathcal{A}$  is weakly amenable and  $\mathcal{A}$  is not ideally amenable.

3- Let  $\mathcal{A}$  be a non-nuclear  $C^*$ -algebra. Then  $\mathcal{A}$  is non-amenable, ideally amenable Banach algebra [E-Y].

4- Let  $\mathcal{A}$  be a commutative non-amenable, weakly amenable Banach algebra. Then  $\mathcal{A}$  is non-amenable,  $n$ -ideally amenable Banach algebra for each  $n \in \mathbb{N}$  [E-Y].

Let  $\mathcal{A}$  be a Banach algebra,  $X$  a Banach  $\mathcal{A}$ -bimodule and  $Y$  a closed  $\mathcal{A}$ -submodule of  $X$ , we say that the short exact sequence  $\{0\} \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \longrightarrow \{0\}$  of  $\mathcal{A}$ -bimodules splits if  $\pi$  has a bounded right inverse which is also an  $\mathcal{A}$ -bimodule homomorphism. The following theorem is well known.

**Theorem 1.1.** Let  $\mathcal{A}$  be a Banach algebra,  $X$  a Banach  $\mathcal{A}$ -bimodule and  $Y$  a closed  $\mathcal{A}$ -submodule of  $X$ . Then the following conditions are equivalent.

- i) The short exact sequence  $\{0\} \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \longrightarrow \{0\}$  splits.
- ii)  $i$  has a bounded left inverse which is also an  $\mathcal{A}$ -bimodule homomorphism.
- iii) There exists a continuous projection of  $X$  onto  $Y$  which is also an  $\mathcal{A}$ -bimodule homomorphism.

See [D] for a proof.

Let  $\mathcal{A}$  be a Banach algebra. Then the projective tensor product of  $\mathcal{A}$  is denoted by  $\mathcal{A} \hat{\otimes}_\pi \mathcal{A}$ . This space is a Banach  $\mathcal{A}$ -bimodule with module multiplications defined by :

$$a.(b \otimes c) = ab \otimes c \quad \text{and} \quad (b \otimes c).a = b \otimes ca \quad (a, b, c \in \mathcal{A}).$$

The corresponding diagonal operator  $\Delta : \mathcal{A} \hat{\otimes}_\pi \mathcal{A} \longrightarrow \mathcal{A}$  is defined by  $a \otimes b \mapsto ab$ . It is clear that  $\Delta$  is a Banach  $\mathcal{A}$ -bimodule homomorphism.

Let  $E$  be a Banach space and  $F(E)$  be the space of finite rank operators on  $E$ . We say that  $E$  has the approximation property if there is a net  $(S_\alpha)_\alpha$  in  $F(E)$  such that  $(S_\alpha) \longrightarrow id_E$  uniformly on compact subsets of  $E$ .

## 2. DERIVATIONS INTO DUALS OF SUBMODULES

Let  $X$  be a Banach  $\mathcal{A}$ -bimodule, and  $Y$  a closed  $\mathcal{A}$ -submodule of  $X$ . By using exact sequences, we give some conditions that  $H^1(\mathcal{A}, X^*) = \{0\}$  implies  $H^1(\mathcal{A}, Y^*) = \{0\}$ .

**Theorem 2.1.** Let  $\mathcal{A}$  be a Banach algebra,  $X$  a Banach  $\mathcal{A}$ -bimodule and  $Y$  a closed  $\mathcal{A}$ -submodule of  $X$ . If  $H^1(\mathcal{A}, X^*) = \{0\}$  and the exact sequence

$$\{0\} \longrightarrow Y^\perp \xrightarrow{i} X^* \xrightarrow{\pi} \frac{X^*}{Y^\perp} \longrightarrow \{0\} \quad (1)$$

of Banach  $\mathcal{A}$ -bimodules splits, then  $H^1(\mathcal{A}, Y^*) = \{0\}$ .

**Proof.** Let  $D : \mathcal{A} \longrightarrow Y^*$  be a derivation. Since the exact sequence (1) splits,  $\pi$  has a bounded right inverse, say  $\phi$ , such that  $\phi$  is also an  $\mathcal{A}$ -bimodule homomorphism. In this case  $\phi \circ D : \mathcal{A} \longrightarrow X^*$  is a derivation, so there exists  $f \in X^*$  such that  $\phi \circ D = \delta_f$ . Thus, we have  $\pi \circ \phi \circ D = \pi \circ \delta_f$ . This shows that  $id_{Y^*} \circ D = \delta_{\pi(f)}$  and therefore  $D = \delta_{\pi(f)}$ . ■

The following lemma is in literature but we give its proof.

**Lemma 2.2.** The exact sequence

$$\{0\} \longrightarrow Y^\perp \xrightarrow{i} X^* \xrightarrow{\pi} \frac{X^*}{Y^\perp} \longrightarrow \{0\} \quad (1)$$

splits, if the following exact sequence splits ;

$$\{0\} \longrightarrow Y \xrightarrow{i} X \xrightarrow{\pi} \frac{X}{Y} \longrightarrow \{0\} \quad (2)$$

**Proof.** Since the exact sequence (2) splits, there exists a continuous projection  $P$  of  $X$  onto  $Y$  which is also an  $\mathcal{A}$ -bimodule homomorphism. Let  $Q = id_{X^*} - P^*$ . Then for each  $y \in Y$  and  $f \in X^*$  we have

$$\begin{aligned} \langle y, Q(f) \rangle &= \langle y, f - P^*f \rangle = \langle y, f \rangle - \langle y, P^*f \rangle \\ &= \langle y, f \rangle - \langle P(y), f \rangle = 0. \end{aligned}$$

So  $Q(X^*) \subseteq Y^\perp$ . On the other hand for each  $f \in X^*$  and  $x \in X$  we have

$$\begin{aligned} \langle x, Q^2(f) \rangle &= \langle x, Q(f - P^*f) \rangle = \langle x, (f - P^*f - P^*(f - P^*f)) \rangle \\ &= \langle x, f \rangle - \langle P(x), f \rangle - \langle P(x), f \rangle + \langle P(x), P^*f \rangle \\ &= \langle x, f \rangle - \langle P(x), f \rangle = \langle x, Q(f) \rangle. \end{aligned}$$

Thus,  $Q$  is a continuous projection of  $X^*$  onto  $Y^\perp$ . Also for  $f \in X^*$ ,  $a \in \mathcal{A}$  and  $x \in X$ , we have

$$\begin{aligned} \langle x, Q(a.f) \rangle &= \langle x, a.f - P^*(a.f) \rangle = \langle x.a, f \rangle - \langle P(x), a.f \rangle \\ &= \langle x.a, f \rangle - \langle P(x).a, f \rangle = \langle x.a, f \rangle - \langle P(x.a), f \rangle \\ &= \langle x.a, Q(f) \rangle = \langle x, a.Q(f) \rangle. \end{aligned}$$

So  $Q$  is a left  $\mathcal{A}$ -module homomorphism. Similarly  $Q$  is a right  $\mathcal{A}$ -module homomorphism and this completes the proof.  $\blacksquare$

**Corollary 2.3.** Let  $\mathcal{A}$ ,  $X$ ,  $Y$  be as in Theorem 2.1. If the exact sequence (2) splits and  $H^1(\mathcal{A}, X^*) = \{0\}$ , then  $H^1(\mathcal{A}, Y^*) = \{0\}$ .

**Corollary 2.4.** Let  $\mathcal{A}$  be a Banach algebra and  $n \in \mathbb{N}$ . If  $H^1(\mathcal{A}, X^{(n+2)}) = \{0\}$ , then  $H^1(\mathcal{A}, X^{(n)}) = \{0\}$ .

**Proof.** Let  $\wedge_{n-1} : X^{(n-1)} \longrightarrow X^{(n+1)}$  be the canonical map. Then the exact sequence

$$\{0\} \longrightarrow X^{(n-1)} \xrightarrow{\wedge_{n-1}} X^{(n+1)} \xrightarrow{\pi} \frac{X^{(n+1)}}{X^{(n-1)}} \longrightarrow \{0\}$$

splits, because the adjoint of  $\wedge_{n-2} : X^{(n+1)} \longrightarrow X^{(n-1)}$ , is a left inverse of  $\wedge_{n-1}$  which is also an  $\mathcal{A}$ -bimodule homomorphism. Now use corollary 2.3.  $\blacksquare$

The next corollary has been proved in [D-Gh-G], but it is an immediate result of Corollary 2.4.

**Corollary 2.5.** Let  $\mathcal{A}$  be  $n+2$ -weakly amenable ( $n+2$ -ideally amenable). Then  $\mathcal{A}$  is  $n$ -weakly amenable ( $n$ -ideally amenable).

### 3. CLOSED IDEALS OF WEAKLY AMENABLE BANACH ALGEBRAS

In this section, we find some closed ideals of a weakly amenable Banach algebra  $\mathcal{A}$  for which  $H^1(\mathcal{A}, I^*)$  is trivial. We denote the linear span of the set  $\{ab : a, b \in \mathcal{A}\}$  by  $\mathcal{A}^2$ . We show that if a closed ideal  $I$  satisfies  $\mathcal{A}^2 \subseteq I$  and  $H^1(\mathcal{A}, I^*) = \{0\}$ , then  $\mathcal{A}^2$  is dense in  $I$ . This is a generalization of Grønbæk's theorem [D, Theorem 2.8.63].

First by Theorem 2.1 and Corollary 2.3 we have the following theorem.

**Theorem 3.1.** Assume that  $\mathcal{A}$  is a weakly amenable Banach algebra. If one of the following conditions holds for each closed ideal  $I$  in  $\mathcal{A}$ , then  $\mathcal{A}$  is ideally amenable.

- i) The exact sequence  $\{0\} \longrightarrow I^\perp \xrightarrow{i} \mathcal{A}^* \xrightarrow{\pi} \frac{\mathcal{A}^*}{I^\perp} \longrightarrow \{0\}$ , splits.
- ii) The exact sequence  $\{0\} \longrightarrow I \xrightarrow{i} \mathcal{A} \xrightarrow{\pi} \frac{\mathcal{A}}{I} \longrightarrow \{0\}$ , splits.

**Theorem 3.2.** Let  $\mathcal{A}$  be a Banach algebra and  $I$  be a closed ideal of  $\mathcal{A}$  with a bounded approximate identity. Then the following conditions are equivalent

- i)  $I$  is weakly amenable;
- ii)  $\mathcal{A}$  is  $I$ -weakly amenable.

**Proof.** Let  $\mathcal{A}$  be  $I$ -weakly amenable and  $D : I \rightarrow I^*$  a derivation. Since  $I$  is pseudo unital  $\mathcal{A}$ -bimodule, by Proposition 2.1.6 of [Run1]  $D$  has an extension  $\bar{D} : \mathcal{A} \rightarrow I^*$  such that  $\bar{D}$  is a derivation. But  $\mathcal{A}$  is  $I$ -weakly amenable, thus  $\bar{D}$  and consequently  $D$  is inner. The converse is

Lemma 2.1 of [E-Y]. ■

We recall that, in a Banach algebra  $\mathcal{A}$ , a net  $(e_\alpha)_\alpha$  is quasi-central if for each element  $a \in \mathcal{A}$  ;  $\lim_\alpha (ae_\alpha - e_\alpha a) = 0$  . Obviously, each approximate identity is a quasi-central net.

**Theorem 3.3.** Let  $\mathcal{A}$  be a weakly amenable Banach algebra and  $I$  a closed ideal of  $\mathcal{A}$  with a quasi-central bounded approximate identity . Then  $\mathcal{A}$  is  $I$ -weakly amenable.

**Proof.** Let  $(e_\alpha)$  be a quasi-central bounded approximate identity in  $I$  and let  $J$  be an ultrafilter on the index set of  $(e_\alpha)$  such that dominates the order filter. Define

$$P : \mathcal{A}^* \longrightarrow \mathcal{A}^*$$

$$\phi \longmapsto w^* - \lim_J (\phi - e_\alpha \cdot \phi)$$

For every  $\phi \in \mathcal{A}^*$  and  $a \in I$  we have

$$\langle a, P\phi \rangle = \lim_J \langle a, \phi - e_\alpha \cdot \phi \rangle = \lim_J \langle a - ae_\alpha, \phi \rangle = 0.$$

Thus  $PA^* \subseteq I^\perp$ . Also for  $\phi \in I^\perp$  and  $a \in \mathcal{A}$ , we have

$$\langle a, P\phi \rangle = \lim_J (\langle a, \phi \rangle - \langle ae_\alpha, \phi \rangle) = \langle a, \phi \rangle.$$

This means that  $P$  is a projection of  $\mathcal{A}^*$  onto  $I^\perp$ . On the other hand for  $a, b \in \mathcal{A}$  and  $\phi \in \mathcal{A}^*$  we have

$$\begin{aligned} \langle b, P(a \cdot \phi) \rangle &= \lim_J \langle b, a \cdot \phi - e_\alpha \cdot (a \cdot \phi) \rangle \\ &= \lim_J \langle ba - be_\alpha a, \phi \rangle = \lim_J \langle ba - bae_\alpha, \phi \rangle \\ &= \lim_J \langle ba, \phi - e_\alpha \cdot \phi \rangle = \langle ba, P(\phi) \rangle \\ &= \langle b, a \cdot P(\phi) \rangle. \end{aligned}$$

So,  $P$  is a left  $\mathcal{A}$ -module homomorphism. Similarly  $P$  is a right  $\mathcal{A}$ -module homomorphism. Therefore the exact sequence  $\{0\} \longrightarrow I^\perp \xrightarrow{i} \mathcal{A}^* \xrightarrow{\pi} \frac{\mathcal{A}^*}{I^\perp} \longrightarrow \{0\}$  splits and consequently  $\mathcal{A}$  is  $I$ -weakly amenable. ■

Let  $\mathcal{A}$  be a Banach algebra and let  $I$  be a closed ideal of  $\mathcal{A}$ . If  $I$  is Arens regular with bounded approximate identity  $(e_\alpha)_\alpha$ , then  $(e_\alpha)_\alpha$  is quasi-central. To prove this, let  $E \in I^{**}$  be a cluster point of  $(e_\alpha)_\alpha$ . Then  $E$  is the identity of  $I^{**}$ . Also for each  $a \in \mathcal{A}$ , we have  $aE, Ea \in I^{**}$  and thus  $aE = E(aE) = (Ea)E = Ea$ .

So, by applying the above theorem, we have the following.

**Corollary 3.4.** Let  $\mathcal{A}$  be a weakly amenable Banach algebra and let  $I$  be a closed ideal of  $\mathcal{A}$  with a bounded approximate identity. If one of the following conditions holds,

- i)  $I$  is Arens regular.

ii)  $\mathcal{A}$  is Arens regular.

Then  $\mathcal{A}$  is  $I$ -weakly amenable.

Since every  $C^*$ -algebra  $\mathcal{A}$  is weakly amenable [Ha], Arens regular, and every closed ideal of  $\mathcal{A}$  has a bounded approximate identity, then for each closed ideal  $I$  of  $\mathcal{A}$ , we have  $H^1(\mathcal{A}, I^*) = \{0\}$ . In the other words, every  $C^*$ -algebra is ideally amenable [E-Y].

**Corollary 3.5.** Let  $\mathcal{A}$  be a weakly amenable Banach algebra such that each closed ideal of  $\mathcal{A}$  has a quasi-central bounded approximate identity. Then  $\mathcal{A}$  is ideally amenable.

Let  $(a_\alpha)_{\alpha \in I}$  be a quasi-central bounded net in  $\mathcal{A}$ . Then the following closed ideals of  $\mathcal{A}$  are called the net ideals of  $\mathcal{A}$  [E].

$$\begin{aligned} I(a_\alpha) &:= \{a \in \mathcal{A} ; \lim_{\alpha} aa_\alpha = a\}, \\ K(a_\alpha) &:= \{a \in \mathcal{A} ; \lim_{\alpha} aa_\alpha = 0\}, \\ C(a_\alpha) &:= \{a \in \mathcal{A} ; (aa_\alpha) \text{ is convergent}\}, \\ L(a_\alpha) &:= \{\lim_{\alpha} aa_\alpha ; a \in C(a_\alpha)\}. \end{aligned}$$

**Theorem.3.6.** Let  $I(a_\alpha), K(a_\alpha), C(a_\alpha)$  and  $L(a_\alpha)$  are as above. If  $L(a_\alpha) = I(a_\alpha)$  and  $\mathcal{A}$  is  $C(a_\alpha)$ -weakly amenable, then

- i)  $\mathcal{A}$  is  $I(a_\alpha)$ -weakly amenable.
- ii)  $\mathcal{A}$  is  $K(a_\alpha)$ -weakly amenable.

**Proof.** Since  $L(a_\alpha) = I(a_\alpha)$ , we have  $C(a_\alpha) = I(a_\alpha) \oplus K(a_\alpha)$  [E]. Consequently the exact sequences  $\{0\} \longrightarrow I(a_\alpha) \longrightarrow C(a_\alpha) \longrightarrow \frac{C(a_\alpha)}{I(a_\alpha)} \longrightarrow \{0\}$  and  $\{0\} \longrightarrow K(a_\alpha) \longrightarrow C(a_\alpha) \longrightarrow \frac{C(a_\alpha)}{K(a_\alpha)} \longrightarrow \{0\}$  split. Now by Corollary 2.3,  $\mathcal{A}$  is both  $I(a_\alpha)$ -weakly amenable and  $K(a_\alpha)$ -weakly amenable. ■

**Theorem.3.7.** Let  $(a_\alpha)_{\alpha \in I}$  be a quasi-central bounded net in  $\mathcal{A}$ . If one of the following assertions holds, then  $\mathcal{A}$  is  $I(a_\alpha)$ -weakly amenable.

- i)  $\mathcal{A}$  is weakly amenable and  $\{a_\alpha : \alpha \in I\} \subseteq I(a_\alpha)$ .
- ii) There exists a codimension one ideal  $M$  of  $\mathcal{A}$  such that  $H^1(M, I(a_\alpha)^*) = \{0\}$ .

**Proof.** If (i) holds, then  $(a_\alpha)_{\alpha \in I}$  is a quasi-central bounded right approximate identity for  $I(a_\alpha)$ . By Theorem 3.3,  $\mathcal{A}$  is  $I(a_\alpha)$ -weakly amenable. Let (ii) holds. Since  $(a_\alpha)_{\alpha \in I}$  is a bounded approximate identity in  $\mathcal{A}$  for  $\mathcal{A}$ -bimodule  $I(a_\alpha)$ , then by Cohen's factorization theorem, we have  $AI(a_\alpha) = I(a_\alpha)\mathcal{A} = I(a_\alpha)$ , so by [G-L, Theorem 2.3],  $H^1(\mathcal{A}, I(a_\alpha)^*) = H^1(M, I(a_\alpha)^*)$ . Thus  $\mathcal{A}$  is  $I(a_\alpha)$ -weakly amenable. ■

We know that if  $\mathcal{A}$  is a weakly amenable Banach algebra, then  $\bar{\mathcal{A}}^2 = \mathcal{A}$  [D, Theorem 2.8.63]. A generalization of this fact is as follows.

**Theorem 3.8.** Let  $\mathcal{A}$  be a Banach algebra and  $\mathcal{A}^2 \subseteq I$  for an arbitrary closed two-sided ideal  $I$  of  $\mathcal{A}$ . If  $\mathcal{A}$  is  $I$ -weakly amenable, then  $\mathcal{A}^2$  is dense in  $I$ .

**Proof.** Assume  $\bar{\mathcal{A}^2} \neq I$ . Then there exists  $0 \neq \varphi \in I^*$  such that  $\varphi|_{\bar{\mathcal{A}^2}} = 0$ . Let  $\Phi \in \mathcal{A}^*$  be a Hahn-Banach extension of  $\varphi$  on  $\mathcal{A}$ . We define  $D : \mathcal{A} \rightarrow I^*$  by  $a \mapsto \langle a, \Phi \rangle \varphi$ . For  $a, b \in \mathcal{A}$ ,  $D(ab) = \langle ab, \Phi \rangle \varphi = 0$ . Also for  $i \in I$  we have  $\langle i, D(a).b \rangle = \langle i, (\langle a, \Phi \rangle \varphi).b \rangle = \langle a, \Phi \rangle \langle bi, \varphi \rangle = 0$ . So  $D(a).b = 0$ . Similarly  $a.D(b) = 0$ . Therefore  $D$  is a derivation from  $\mathcal{A}$  into  $I^*$  and by hypothesis there exists  $\psi \in I^*$  such that  $D = \delta_\psi$ . Also for every  $i \in I$  we have

$$\begin{aligned} \langle i, \varphi \rangle^2 &= \langle i, \varphi \rangle \langle i, \Phi \rangle = \langle i, D(i) \rangle \\ &= \langle i, \delta_\psi(i) \rangle = \langle i, i\psi - \psi i \rangle \\ &= 0. \end{aligned}$$

So  $\varphi = 0$ , which is contradiction. Thus  $\mathcal{A}^2$  is dense in  $I$ . ■

**Corollary 3.9.** Let  $\mathcal{A}$  be a Banach algebra and let  $M$  be a closed non maximal modular ideal of  $\mathcal{A}$  with codimension one. If  $H^1(\mathcal{A}, M^*) = \{0\}$ , then  $\mathcal{A}^2$  is dense in  $M$ .

**Proof.** Since the codimension of  $M$  is one, there exists  $a \in \mathcal{A}$  such that  $\mathcal{A} = \mathbb{C}a + M$ . We show that  $a^2 \in M$ . Assume that  $a^2$  does not belong to  $M$ , so there exists  $0 \neq \alpha$  and  $m \in M$  such that  $a^2 = \alpha a + m$ . Now let  $b$  be an arbitrary element of  $\mathcal{A}$ , there exist  $\beta$  and  $m' \in M$  such that  $b = \beta a + m'$ , so

$$\begin{aligned} b - b(\alpha^{-1}a) &= (\beta a + m') - (\beta a + m')(\alpha^{-1}a) \\ &= m' - \beta \alpha^{-1}m - \alpha^{-1}m'a. \end{aligned}$$

Thus, for each  $b$ ,  $b - b(\alpha^{-1}a)$  belongs to  $M$ . This means that  $\alpha^{-1}a$  is a left modular identity for  $M$ . Similarly  $\alpha^{-1}a$  is a right modular identity for  $M$ , so  $M$  is a maximal modular ideal which is contradiction. Therefore  $a^2$  belongs to  $M$  and consequently  $\mathcal{A}^2 \subseteq M$ . Now by the above theorem,  $\mathcal{A}^2$  is dense in  $M$ . ■

#### 4. CLOSED IDEALS OF BIFLAT BANACH ALGEBRAS

We say that a Banach algebra  $\mathcal{A}$  is biprojective if  $\Delta : \mathcal{A} \hat{\otimes}_\pi \mathcal{A} \rightarrow \mathcal{A}$  has a bounded right inverse which is an  $\mathcal{A}$ -bimodule homomorphism. Also we say that a Banach algebra  $\mathcal{A}$  is biflat if the bounded linear map  $\Delta^* : \mathcal{A}^* \rightarrow (\mathcal{A} \hat{\otimes}_\pi \mathcal{A})^*$  has a bounded left inverse which is an  $\mathcal{A}$ -bimodule homomorphism [Run1]. Obviously by taking adjoints, one sees that every biprojective Banach algebra is biflat. It is well known that every biflat Banach algebra is weakly amenable [D], and a Banach algebra is amenable if and only if it is biflat and has a bounded approximate identity [Run2].

Since there is no Hahn-Banach theorem for operators, there is none for bilinear continuous forms. In other words, let  $E$  and  $F$  be two Banach spaces and  $G$  be a subspace of  $E$  and  $\phi \in BL(G, F; \mathbb{C})$ , where  $BL(G, F; \mathbb{C})$  is the set of all bounded bilinear mappings from  $G \times F$  into  $\mathbb{C}$ . In general case, there is no any extension of  $\phi$  to a bilinear map  $\tilde{\phi} \in BL(E, F; \mathbb{C})$ . Since  $BL(G, F; \mathbb{C}) \simeq L(G, F^*)$  this situation is equivalent to say that each element  $T \in L(G, F^*)$  doesn't have any extension to an element  $\tilde{T} \in L(E, F^*)$  [D-F,1.5].

However, there is some conditions that Hahn-Banach theorem works for operators as well. Let  $\pi(z; E, F)$  be the projective norm of the element  $z \in F \hat{\otimes}_\pi F$  and  $G$  be a subspace of  $E$ . Then it is clear that  $\pi(z; E, F) \leq \pi(z; G, F)$  for each element  $z \in G \hat{\otimes}_\pi F$ . If there exists  $\lambda \geq 1$  such that  $\pi(z; G, F) \leq \lambda \pi(z; E, F)$  for each element  $z \in G \hat{\otimes}_\pi F$ , then we say that  $\hat{\otimes}_\pi F$  respects  $G$  into  $E \hat{\otimes}_\pi F$  isomorphically. For example,  $\hat{\otimes}_\pi F$  respects  $G$  into  $E \hat{\otimes}_\pi F$  isomorphically if  $G$  is a complemented subspace of  $E$  [D-F].

By Hahn-Banach theorem we can extend each element of  $T \in (G \hat{\otimes}_\pi F)^*$  to a continuous linear functional  $\tilde{T}$  on  $E \hat{\otimes}_\pi F$  provided that  $\hat{\otimes}_\pi F$  respects  $G$  into  $E \hat{\otimes}_\pi F$  isomorphically [D-F].

Now, we are ready to bring our main theorem about biflat Banach algebras. Before doing this we recall that an ideal  $I$  is left essential as a left Banach  $\mathcal{A}$ -module if the linear span of  $\{ai : a \in \mathcal{A}, i \in I\}$  is dense in  $I$ .

**Theorem.4.1.** Let  $\mathcal{A}$  be a biflat Banach algebra and  $I$  a closed ideal of  $\mathcal{A}$  which is left essential. If  $\mathcal{A} \hat{\otimes}_\pi$  respects  $I$  into  $\mathcal{A} \hat{\otimes}_\pi \mathcal{A}$  isomorphically, then  $H^1(\mathcal{A}, I^*) = \{0\}$ .

**Proof.** Let  $D : \mathcal{A} \longrightarrow I^*$  be a derivation. Since  $\mathcal{A}$  is biflat,  $\Delta^* : \mathcal{A}^* \longrightarrow (\mathcal{A} \hat{\otimes}_\pi \mathcal{A})^*$  has a bounded left inverse  $\rho$  which is an  $\mathcal{A}$ -bimodule homomorphism. Let

$$\begin{aligned} T : L(\mathcal{A}, I^*) &\longrightarrow (\mathcal{A} \hat{\otimes}_\pi I)^* \\ S &\longmapsto T_S \end{aligned}$$

be the isometric isomorphism which is defined by  $\langle a \otimes i, T_S \rangle = \langle i, S(a) \rangle$ . Let  $\tilde{T}_D$  be a Hahn-Banach extension of  $T_D$  on  $\mathcal{A} \hat{\otimes}_\pi \mathcal{A}$ . We claim that  $D = \delta_\phi$ , where  $\phi = \rho(\tilde{T}_D)|_I$ .

First we show that for each  $i \in I$  and  $a \in \mathcal{A}$ ;  $i.(a.\tilde{T}_D - \tilde{T}_D.a) = i.\Delta^*(\widetilde{Da})$ , where  $\widetilde{Da}$  is a

Hahn-Banach extension of  $Da$  on  $\mathcal{A}$ . Let  $b, c \in \mathcal{A}$ , we have

$$\begin{aligned}
\langle b \otimes c, i.(a.\tilde{T}_D - \tilde{T}_D.a) \rangle &= \langle b \otimes ci, a.\tilde{T}_D - \tilde{T}_D.a \rangle \\
&= \langle b \otimes cia - ab \otimes ci, \tilde{T}_D \rangle \\
&= \langle b \otimes cia, T_D \rangle - \langle ab \otimes ci, T_D \rangle \\
&= \langle cia, Db \rangle - \langle ci, D(ab) \rangle \\
&= \langle ci, a.Db - D(ab) \rangle = \langle ci, Da.b \rangle \\
&= \langle bci, Da \rangle = \langle bci, \tilde{Da} \rangle \\
&= \langle b \otimes ci, \Delta^*(\tilde{Da}) \rangle \\
&= \langle b \otimes c, i. \Delta^*(\tilde{Da}) \rangle.
\end{aligned}$$

Now, let  $a, b \in \mathcal{A}$  and  $i \in I$ . Then we have

$$\begin{aligned}
\langle bi, \delta_\phi(a) \rangle &= \langle bi, a.\phi - \phi.a \rangle \\
&= \langle bi, a.\rho(\tilde{T}_D)|_I - \rho(\tilde{T}_D)|_I.a \rangle \\
&= \langle bia - abi, \rho(\tilde{T}_D)|_I \rangle \\
&= \langle bia - abi, \rho(\tilde{T}_D) \rangle \\
&= \langle bi, a.\rho(\tilde{T}_D) - \rho(\tilde{T}_D).a \rangle \\
&= \langle b, \rho(i.(a.\tilde{T}_D - \tilde{T}_D.a)) \rangle \\
&= \langle b, \rho(i. \Delta^*(\tilde{Da})) \rangle \\
&= \langle bi, id_{\mathcal{A}^*}(\tilde{Da}) \rangle.
\end{aligned}$$

Since  $I$  is left essential and  $Da, \delta_\phi(a)$  are both continuous linear functional on  $I$ , we have  $Da = \delta_\phi(a)$ . This is true for each  $a \in \mathcal{A}$ , so  $D = \delta_\phi$  and  $D$  is inner.  $\blacksquare$

**Corollary.4.2.** Let  $\mathcal{A}$  be a biflat Banach algebra with a left approximate identity. Then  $\mathcal{A}$  is ideally amenable provided that  $\mathcal{A} \hat{\otimes}_\pi \cdot$  respect all closed ideals into  $\mathcal{A} \hat{\otimes}_\pi \mathcal{A}$  isomorphically.

There are a kind of biprojective Banach algebras whose left closed ideals are left essential. These algebras are semiprime biprojective Banach algebras with the approximation property [S].

**Corollary.4.3.** Let  $\mathcal{A}$  be a semiprime, biprojective Banach algebra with the approximation property, and  $I$  a closed ideal of  $\mathcal{A}$ . If  $\mathcal{A} \hat{\otimes}_\pi \cdot$  respects  $I$  into  $\mathcal{A} \hat{\otimes}_\pi \mathcal{A}$  isomorphically, then  $H^1(\mathcal{A}, I^*) = \{0\}$ . In particular, for each closed ideal  $I$  which is complemented as a subspace of  $\mathcal{A}$ , the assertion holds.

## 5. SOME HEREDITARY PROPERTIES OF IDEAL AMENABILITY

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two Banach algebras and  $\phi : \mathcal{A} \longrightarrow \mathcal{B}$  a continuous homomorphism with dense range. We know that  $\mathcal{B}$  is amenable if  $\mathcal{A}$  is amenable [J1], but this is not true for weak amenability. In special case, if  $\mathcal{A}$  is weakly amenable and commutative, then  $\mathcal{B}$  is weakly amenable [D].

**Theorem 5.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  are two Banach algebras and let  $\phi : \mathcal{A} \longrightarrow \mathcal{B}$  be a continuous homomorphism with dense range and  $J$  a closed ideal of  $\mathcal{B}$ . If the following conditions hold:

- i)  $\phi|_{J^c}$  is one to one, where  $J^c$  is  $\phi^{-1}(J)$ .
- ii)  $\phi(J^c)$  is dense in  $J$ .
- iii)  $H^1(\mathcal{A}, J^{c*}) = \{0\}$ .

Then  $H^1(\mathcal{B}, J^*) = \{0\}$ .

**Proof.** Let  $D : \mathcal{B} \longrightarrow J^*$  be a derivation. Define  $T : J^* \longrightarrow J^{c*}$ , given by  $f \mapsto T_f$ , where  $T_f$  is defined by  $T_f(a) = f(\phi(a))$ . Obviously for each  $f \in J^*$ ,  $T_f$  is a continuous linear functional on  $J^c$  and so  $T$  is well defined. We show that  $T$  is onto.

Let  $g \in J^{c*}$  and define  $\tilde{f} : \phi(J^c) \longrightarrow \mathbb{C}$  by  $\phi(a) \mapsto g(a)$ .  $\phi(J^c)$  is a subspace of  $J$  and  $\tilde{f}$  is a bounded linear functional on  $\phi(J^c)$ . By Hahn-Banach theorem, there exists a continuous linear functional  $f$  on  $J$  such that  $f|_{\phi(J^c)} = \tilde{f}$ . It is clear that  $T_f = g$ .

Now define  $\tilde{D} : \mathcal{A} \longrightarrow J^{c*}$  by  $\tilde{D} := T \circ D \circ \phi$ .  $\tilde{D}$  is a bounded linear map. Let  $a_1, a_2 \in \mathcal{A}$  and  $a \in J^c$ . Then

$$\begin{aligned}
 \langle \tilde{D}(a_1 a_2), a \rangle &= \langle T(D(\phi(a_1 a_2))), a \rangle \\
 &= \langle T(D(\phi(a_1)) \cdot \phi(a_2) + \phi(a_1) \cdot D(\phi(a_2))), a \rangle \\
 &= \langle D(\phi(a_1)) \cdot \phi(a_2) + \phi(a_1) \cdot D(\phi(a_2)), \phi(a) \rangle \\
 &= \langle D(\phi(a_1)), \phi(a_2 a) \rangle + \langle D(\phi(a_2)), \phi(a a_1) \rangle \\
 &= \langle T \circ D \circ \phi(a_1), a_2 a \rangle + \langle T \circ D \circ \phi(a_2), a a_1 \rangle \\
 &= \langle \tilde{D}(a_1) \cdot a_2, a \rangle + \langle a_1 \cdot \tilde{D}(a_2), a \rangle
 \end{aligned}$$

Thus  $\tilde{D}$  is a derivation. Since  $H^1(\mathcal{A}, J^{c*}) = \{0\}$ , then there exists  $g \in J^{c*}$  such that  $\tilde{D} = \delta_g$ . But  $T$  was onto so, there exists  $f \in J^*$  such that  $T_f = g$ . We claim that  $D = \delta_f$ . Let  $a \in \mathcal{A}$  and

$a_1 \in J^c$ , We have

$$\begin{aligned}
\langle D(\phi(a)), \phi(a_1) \rangle &= \langle \tilde{D}(a), a_1 \rangle = \langle a.g - g.a, a_1 \rangle \\
&= \langle g, a_1 a - a a_1 \rangle \\
&= \langle T_f, a_1 a - a a_1 \rangle \\
&= \langle f, \phi(a_1 a - a a_1) \rangle \\
&= \langle \phi(a).f - f.\phi(a), \phi(a_1) \rangle \\
&= \langle \delta_f(\phi(a)), \phi(a_1) \rangle
\end{aligned}$$

Since  $\phi(J^c)$  is dense in  $J$ , we have  $D(\phi(a)) = \delta_f(\phi(a))$  for each  $a \in \mathcal{A}$ . Again since  $\phi(\mathcal{A})$  is dense in  $\mathcal{B}$ , then  $D = \delta_f$ . ■

Let  $\mathcal{A}$  and  $\mathcal{B}$  be two Banach algebras and  $\phi : \mathcal{A} \longrightarrow \mathcal{B}$  a continuous homomorphism with dense range. In general, we assert that the ideal amenability of  $\mathcal{A}$  doesn't imply the ideal amenability of  $\mathcal{B}$ .

We know that, the approximation property is not necessary for the weak amenability of the algebra of approximable operators on a Banach space [B, Corollary.3.5]. Also there are some Banach spaces  $E$  with the approximation property such that  $A(E)$  is not weakly amenable [B, Theorem.5.3].

Now let  $E$  be a Banach space with the approximation property such that  $A(E)$  is not weakly amenable. Then the nuclear algebra  $N(E)$  of  $E$  is biprojective and consequently weakly amenable. Since  $N(E)$  is topologically simple, then  $N(E)$  is ideally amenable. On the other hand,  $A(E)$  is not ideally amenable and the inclusion map  $i : N(E) \longrightarrow A(E)$  is a continuous homomorphism with dense range. This proves the assertion.

**Theorem.5.2.** Suppose  $Y$  and  $Z$  are closed subspaces of a Banach space  $X$ , and suppose that there is a collection  $\Lambda \subset B(X)$  with the following properties:

- i) Every  $\phi \in \Lambda$  maps  $X$  into  $Y$ .
- ii) Every  $\phi \in \Lambda$  maps  $Z$  into  $Z$ .
- iii)  $\sup\{\|\phi\| : \phi \in \Lambda\} < \infty$ .
- iv) To every  $y \in Y$  and to every  $\epsilon > 0$  corresponds a  $\phi \in \Lambda$  such that  $\|y - \phi y\| < \epsilon$ .

Then  $Y + Z$  is closed.

**Proof.** [Rud, 1.2.Theorem].

**Corollary.5.3.** Suppose  $\mathcal{A}$  is a Banach algebra. Let  $I$  be a right closed ideal and  $J$  be a left closed ideal of  $\mathcal{A}$ . If  $I$  has a bounded approximate identity, then  $I + J$  is closed.

**Proof.** Let  $(e_\alpha)_\alpha$  be a bounded approximate identity for  $I$  and

$$\Lambda = \{L_{e_\alpha} : \mathcal{A} \longrightarrow \mathcal{A} \mid L_{e_\alpha}(a) = e_\alpha a \}.$$

Obviously  $\Lambda \subset B(\mathcal{A})$ . For each  $\alpha$ , we have  $L_{e_\alpha}(\mathcal{A}) = \{e_\alpha a : a \in \mathcal{A}\} \subset I$  and  $L_{e_\alpha}(J) = \{e_\alpha j : j \in J\} \subset J$ . Also we have

$$\sup\{\|L_{e_\alpha}\| : L_{e_\alpha} \in \Lambda\} \leq \sup_\alpha\{\|e_\alpha\|\} < \infty.$$

Now let  $\epsilon > 0$  is given and  $i \in I$ . There exists  $\alpha_0$  such that

$$\|i - L_{e_{\alpha_0}}(i)\| = \|i - e_{\alpha_0}i\| < \epsilon.$$

Thus by the above theorem,  $I + J$  is closed. ■

**Theorem.5.4.** Let  $\mathcal{A}$  be a Banach algebra and let  $I$  be a closed ideal of  $\mathcal{A}$  with bounded approximate identity. If  $I$  and  $\frac{\mathcal{A}}{I}$  are ideally amenable, then  $\mathcal{A}$  is ideally amenable.

**Proof.** Let  $J$  be a closed ideal of  $\mathcal{A}$  and  $D : \mathcal{A} \longrightarrow J^*$  a derivation. Consider  $\iota : I \cap J \longrightarrow J$  as an inclusion map. Obviously  $\iota^* : J^* \longrightarrow (I \cap J)^*$  is an  $\mathcal{A}$ -bimodule homomorphism and so,  $\iota^* \circ D : \mathcal{A} \longrightarrow (I \cap J)^*$  and consequently  $\iota^* \circ D|_I : I \longrightarrow (I \cap J)^*$  is a continuous derivation. Since  $I$  is ideally amenable, there exists  $\phi_1 \in (I \cap J)^*$  such that  $\iota^* \circ D|_I = \delta_{\phi_1}$ . Let  $\Phi_1$  be the Hahn-Banach extension of  $\phi_1$  on  $J$ . Define  $\tilde{D} := D - \delta_{\Phi_1}$ . So,  $\tilde{D}$  is a derivation from  $\mathcal{A}$  into  $J^*$ . We show that  $\tilde{D}|_I = 0$ . Let  $i \in I$  and  $j \in J$ , we have

$$\begin{aligned} \langle j, \tilde{D}(i) \rangle &= \langle j, D(i) \rangle - \langle j, \delta_{\Phi_1}(i) \rangle \\ &= \langle j, D(i) \rangle - \langle ji - ij, \Phi_1 \rangle \\ &= \langle j, D(i) \rangle - \langle ji - ij, \phi_1 \rangle \\ &= \langle j, D(i) \rangle - \langle j, \delta_{\phi_1}(i) \rangle \\ &= \langle j, D(i) \rangle - \langle j, \iota^*(D(i)) \rangle \\ &= \langle j, D(i) \rangle - \langle \iota(j), D(i) \rangle = 0. \end{aligned}$$

So,  $\tilde{D}|_I = 0$  and  $\tilde{D}$  induces a map from  $\frac{\mathcal{A}}{I}$  into  $J^*$ , we call it  $\tilde{D}$  itself, which is a derivation. Since  $I \subset \text{Ann}(\frac{J}{J \cap I})$ , the annihilator of  $(\frac{J}{J \cap I})$ , then  $\frac{J}{J \cap I}$  is a Banach  $\frac{\mathcal{A}}{I}$ -bimodule.

On the other hand, let  $(e_\alpha)_\alpha$  be a bounded approximate identity in  $I$ . Then for each  $x \in I \cap J$  and  $a \in \mathcal{A}$  we have

$$\begin{aligned} \langle x, \tilde{D}(a + I) \rangle &= \lim_\alpha \langle xe_\alpha, \tilde{D}(a) \rangle \\ &= \lim_\alpha \langle e_\alpha, \tilde{D}(a).x \rangle \\ &= \lim_\alpha \langle e_\alpha, \tilde{D}(ax) - a.\tilde{D}(x) \rangle \\ &= 0. \end{aligned}$$

So,  $\tilde{D}(\frac{\mathcal{A}}{I}) \subseteq (I \cap J)^\perp \cong (\frac{J}{J \cap I})^*$ . Since  $I$  has a bounded approximate identity, then by Corollary 5.3,  $J + I$  is a closed ideal of  $\mathcal{A}$ , thus  $\frac{J+I}{I}$  is a Banach space. Now define  $\psi : \frac{J}{J \cap I} \longrightarrow \frac{J+I}{I}$

by  $j + J \cap I \mapsto j + I$ . Obviously  $\psi$  is an algebra isomorphism. Therefore  $\psi$  is an  $\frac{\mathcal{A}}{I}$ -bimodule isomorphism. Also

$$\begin{aligned} \|\psi(j + J \cap I)\| &= \|j + I\| \\ &= \inf\{\|j + i\| : i \in I\} \\ &\leq \inf\{\|j + i\| : i \in I \cap J\} \\ &= \|j + J \cap I\|. \end{aligned}$$

So,  $\psi$  is bounded. By open mapping theorem  $\psi$  is a homeomorphism and consequently  $(\frac{J}{J \cap I})^* \cong (\frac{J+I}{I})^*$ . Therefore there exists  $\Phi_2 \in (I \cap J)^\perp$  such that  $\tilde{D} = \delta_{\Phi_2}$ . It shows that  $D = \delta_{\Phi_1 + \Phi_2}$ ;  $D$  is inner and  $\mathcal{A}$  is ideally amenable. ■

It is notable that the above theorem is true for weak amenability and amenability even if  $I$  does not have any bounded approximate identity [D].

Now we pose an open problem in this direction.

**Question.** Is valid the above theorem, if  $I$  does not have any bounded approximate identity ?

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